



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: March 27, 2014

ACCEPTED: May 8, 2014

PUBLISHED: June 3, 2014

Non-integrability in non-relativistic theories

Dimitrios Giataganas^{a,b} and Konstadinos Sfetsos^{a,c}^a*Department of Nuclear and Particle Physics, Faculty of Physics,
University of Athens, Athens 15784, Greece*^b*National Institute for Theoretical Physics,
School of Physics and Centre for Theoretical Physics,
University of the Witwatersrand, Wits, 2050, South Africa*^c*Department of Mathematics, University of Surrey,
Guildford GU2 7XH, U.K.**E-mail:* dgiataganas@phys.uoa.gr, ksfetsos@phys.uoa.gr

ABSTRACT: Generic non-relativistic theories giving rise to non-integrable string solutions are classified. Our analysis boils down to a simple algebraic condition for the scaling parameters of the metric. Particular cases are the Lifshitz and the anisotropic Lifshitz spacetimes, for which we find that for trivial dilaton dependence the only integrable physical theory is that for $z = 1$. For the hyperscaling violation theories we conclude that the vast majority of theories are non-integrable, while only for a small class of physical theories, where the Fermi surfaces belong to, integrability is not excluded. Schrödinger theories are also analyzed and a necessary condition for non-integrability is found. Our analysis is also applied to cases where the exponential of the dilaton is a monomial of the holographic coordinate.

KEYWORDS: Gauge-gravity correspondence, AdS-CFT Correspondence, Integrable Field Theories

ARXIV EPRINT: [1403.2703](https://arxiv.org/abs/1403.2703)

Contents

1	Introduction and method	1
2	Classes of theories we focus on	4
3	Non-integrability in generic non-relativistic spaces	6
3.1	Consistency of sigma-model solutions	6
3.2	Non-integrability of the string solutions	7
3.3	Application to theories of special interest	9
3.3.1	Lifshitz non-integrability	9
3.3.2	Anisotropic Lifshitz-Like non-integrability	10
3.3.3	Hyperscaling violation metrics non-integrability	10
4	Geometries with non-diagonal terms	11
4.1	Application to Schrödinger geometries	12
5	Conclusions	13

1 Introduction and method

In recent years there has been intensive research concerning the integrability of string theoretical models in curved space. The strings are described by two-dimensional sigma-models and their classical motion is constrained by the equations of motion which typically form a non-linear system of differential equations. Among the plethora of models some are of special interest due to their integrable structure.

In principle, in order to show that a system is integrable one has to demonstrate that there are as many integrals of motion as degrees of freedom. For the cases of sigma-models that we are mostly interested in, the standard way to show integrability is to find the Lax pair representation, from which an infinite number of conserved charges follow. In an AdS/CFT context one equivalently derives a system of integral string Bethe equations which have been proven equivalent to the thermodynamic limit of the spin chain Bethe equations. In this way the matching of the multi-spin string energies to the conformal dimension of the field theory operators is possible [1–3].

The full set of necessary conditions for the existence of a Lax pair is not known. Considering this, it is natural to think of other ways to prove or, for that matter, disprove integrability of a system. In an integrable two-dimensional sigma-model all subsystems that one can derive by consistent truncations should also be integrable. Hence, a necessary condition for a two-dimensional sigma-model to be integrable is that when the system of second order partial differential equations is consistently reduced to a system of ordinary

differential equations, then this has to constitute an integrable system. Similarly, one can prove non-integrability by demonstrating that at least one consistent one-dimensional truncation of the full system of equations of motion is a non-integrable system of second-order differential equations. This can be done either through numerical methods in chaos, or through semi-analytic algorithmic approaches [4, 5] and [6–8].

These methods of proving non-integrability have been applied so far to cosmological models as well as in the context of the gauge/gravity correspondence. Signs of chaotic behavior have been found in Schwarzschild black holes [9, 10]. In the confining AdS soliton background [11] the effective Lagrangian of string solutions was found to be reduced to a set of coupled harmonic and unharmonic oscillators where at high energies result in a chaotic and non-integrable behavior [12]. Moreover, the class of the Sasaki-Einstein manifolds was tested starting with the particular example of the conifold $AdS_5 \times T^{1,1}$ and later for $Y^{p,q}$ manifolds [13, 14]. The string solutions again show chaotic behavior implying nonintegrability of the corresponding sigma models. However, in $Y^{p,q}$ manifolds special care should be taken with the string solution to show that indeed they satisfy the Sasaki-Einstein constraints as in [15–17]. More recently it has been shown that integrability is not present in marginal complex β -deformations [18]. In addition, working with non-integrable geodesics it has been shown that integrability is ruled out for $\mathcal{N} = 4$ SYM beyond the planar limit [19], although at special large N limits some integrability does appear [20, 21]. Furthermore, the classical spinning solutions have been found to have chaotic behavior in various confining backgrounds: Klebanov-Strassler, Maldacena-Nunez, Witten AdS and the AdS soliton [22]. It is natural to think therefore that confining backgrounds have a chaotic quantum spectrum in the sense of the Gaussian orthogonal ensemble (GOE) [23]. The quantum spectrum has also been studied in the AdS soliton background where a smooth transition from the Wigner GOE to the integrable regime was noticed as the energy increased, in agreement with the fact that the system becomes asymptotically AdS at higher energies [24]. In [25] the string solutions were studied in curved Dp-brane backgrounds and non-integrability was found except for special cases. In [26] the response of a nonlinearly coupled scalar field in the probe limit of an asymptotically AdS black brane geometry was studied and it was argued that the response of the dual operator is chaotic. Similarly in [27], the response of the scalar field to the gravitational collapse in an asymptotically Anti de Sitter space-time was found to be chaotic.

In this paper we extend these studies to the non-relativistic gauge/gravity dualities. There are many interesting systems which have general dynamical scalings instead of scale invariance, as for example the Lifshitz, the spatial anisotropic Lifshitz-like and the Schrödinger spaces, as well as backgrounds with hyperscaling violations of the above. General Lifshitz geometries may be constructed, in a phenomenological way, by switching on fluxes with non-trivial topological couplings [28]. Explicit Lifshitz solutions in string theory are more difficult to construct. Progress in that direction has been made in [29, 30] and in [31] where it has been proved that Lifshitz solutions with dynamical exponent $z = 2$ can be embedded in supergravity. Later in [32] explicit type-II supergravity solutions of the form $Lif_p \times \Omega_{d-p}$ with generic dynamical exponents, where Ω_{d-p} is a constant curvature spherical flat or hyperbolic space, have been found. In the meantime anisotropic

Lifshitz-like solutions were found in [33] and in addition they have been extended to finite temperature cases [34] and also used to study anisotropic plasmas, eg. [35, 36]. The non-relativistic conformal field theories can also be constructed as cosets of the Schrödinger group and its variants as in [37]. Moreover, it has been found that the IR geometry of a dilaton theory with an asymptotic electric flux is Lifshitz [38]. Other non-relativistic constructions have been found in [39].

Motivated by all these developments we consider the integrability of string solutions in non-relativistic theories. We find generic conditions in terms of the scaling parameters of the metric elements of the backgrounds, which when not satisfied imply non-integrability. For the particular example of the Lifshitz backgrounds with trivial dilaton dependence we find that the only integrable solution is the case with $z = 1$, corresponding to the AdS space. When the hyperscaling violation scaling is considered, only for a subset of all possible backgrounds integrability is not excluded. This subset includes the Fermi surfaces. We derive explicitly the bounds of the non-integrable physical theories. We additionally analyze spaces with non-diagonal metric terms, where the Schrodinger metrics are particular examples and we find cases that integrability is excluded.

It is interesting that for the cases we find non-integrability, there is no sign of chaos inside the accepted range of the parameters and coordinates and when satisfying the Virasoro constraints. We checked this for solutions obtained from a wide choice of the initial values in the system of differential equations. Notice that it does not necessarily mean that for other energies, not fixed by the Virasoro constrain, or other string configurations the system is not chaotic. The integrability of dynamical systems is related on how the system reacts to small variations around its phase space curves. According to the Kolmogorov-Arnold-Moser (KAM) theorem, when the system is non-integrable it is resonant in the corresponding phase space describing the motion in the angle variables. Practically, when weak non-linear perturbations are applied to an integrable Hamiltonian system whose motion is confined to invariant tori, then some deformed invariant tori may still be present and some others may be destroyed. One would like to know, whether or not the perturbed solutions are stable and if the perturbed orbits will remain close to unperturbed ones for long period of time. The conditions for the regular and normal behavior of the perturbed solution are provided by the KAM theorem. Chaos can occur only when the KAM theorem does not hold. Chaotic motion arises due to some appropriate noise applied to usually well behaved solutions, resulting to a motion that is highly sensitive to initial conditions. Moreover, the appearance of chaos can also be viewed as a breakdown of integrability, but non-integrability does not always imply chaotic behavior.

The chaotic motion can be realized easier from the Poincare maps. By fixing the energy of the solutions of the Hamiltonian system and choosing one invariant plane, one can obtain curves on a two dimensional plane generated by fixing the initial conditions. When the curves are in a sense continuous, the solutions are regular. These regular trajectories are obtained by integrating the equations of system for the particular conditions, and the solutions are unique and reproducible. When individual points appear, the chaotic regions become apparent. These points are located randomly and integrability breaks down over the phase space. There is a possibility to have regions of phase space which

show integrability and regions of chaos for different initial conditions. For our solutions we have not found regions of chaotic motion for the particular fixed energy of the system imposed by the zero Hamiltonian. This does not mean that the system is not chaotic, since in other energies chaos may occur.

Alternatively, to the aforementioned techniques, by using differential Galois theory, it can be examined if the identity component of the differential Galois group of variational equations normal to an integrable plane of solutions is Abelian. However, in practice, in order to determine the Galois group is a difficult task. Therefore, one takes advantage of the fact that the Abelian identity element in the Galois group is equivalent to finding Liouvillian solutions, i.e. combinations of algebraic functions, exponentiation of quadratures and quadratures, to the normal variational equation. This is in principle an easier task, and for the Hamiltonian systems we are interested in, we can end up with second order homogeneous differential equations with rational coefficients, where the existence of the Liouvillian solutions, can be checked with the running of the Kovacic algorithm [40], which only fails when the system has no Liouvillian solutions.

The plan of the paper is as follows: in the section 2 we introduce the theories of special interest we plan to examine as particular cases of the generic analysis. In section 3, we start by working with a generic sigma model. Then we study the integrability of string configurations in general non-relativistic spaces and apply our results to backgrounds of special interest. In section 4, we derive the conditions of non-integrability for metrics with non-diagonal terms and apply them to known backgrounds. In section 5, we conclude and discuss our results and further directions.

2 Classes of theories we focus on

In this section we give a brief overview of the properties of a class of backgrounds of special interest which are particular examples of the generic study in the subsequent sections.

Let us consider $(d + 2)$ -dimensional Lifshitz metrics of the form

$$ds^2 = r^{-2\theta/d} \left(-r^{2z} dt^2 + r^2 \sum_{i=1}^d dx_i^2 + \frac{dr^2}{r^2} \right), \quad (2.1)$$

where the constants z and θ are the dynamical critical and hyperscaling violation exponents, respectively. This metric transforms covariantly under the scaling transformation

$$t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i, \quad r \rightarrow r/\lambda, \quad ds \rightarrow \lambda^{\theta/d} ds. \quad (2.2)$$

In the $SO(d)$ -invariant plane spanned by the coordinates x_i , $i = 1, 2, \dots, d$, we single out the 1–2 plane in which we use polar coordinates (ρ, ϕ) .

We plan to study the non-integrability of such holographic theories, although our results may apply widely in theories of general gravitational interest. In addition, we will consider the restrictions imposed in the present context by the null energy condition

$$T_{\mu\nu} N^\mu N^\nu \geq 0, \quad (2.3)$$

in order for the dual field theory to be physically sensible [41]. This implies that

$$(d - \theta)(dz - \theta - d) \geq 0, \quad (z - 1)(d + z - \theta) \geq 0. \quad (2.4)$$

For $\theta = 0$ we recover the Lifshitz space Lif_z physical constraint with the condition $z \geq 1$. The saturation of the bound happens for $z = 1$ where the spacetime becomes AdS_{d+2} . The exponent θ is holographically related to the thermal entropy density in the dual theory which scales as $S \simeq T^{(d-\theta)/z}$.

An additional physical requirement for any local quantum field theory is that the entanglement entropy of a given region scales with the area of its boundary up to logarithmic corrections. This gives the condition [42]

$$\theta \leq d - 1, \quad (2.5)$$

which also excludes non-stable theories. It has been shown that when this bound is saturated for $\theta = d - 1$, the dual theory exhibits a Fermi surface [42, 43]. Then the condition (2.4) simplifies to $z \geq 2 - 1/d$. When this is also saturated, the corresponding values of the scaling parameters are of particular interest, since they can be related to the non-Fermi liquid states, for example the $d = 2$, giving $\theta = 1$ and $z = 3/2$ [43].

Our discussion can be generalized to anisotropic Lifshitz-like theories. For vanishing hyperscaling parameter θ the corresponding metric takes the form

$$ds^2 = r^{2z} \left(-dt^2 + \sum_{i=1}^k dy_i^2 \right) + r^2 \sum_{i=k+1}^d dx_i^2 + \frac{dr^2}{r^2}, \quad (2.6)$$

for which the $SO(d)$ symmetry of the spatial part is broken down to $SO(k) \times SO(d - k)$. In this case the parameter z measures not only the degree of Lorentz symmetry violation, but also the anisotropy of the full space spanned by all the y_i 's and x_i 's. This metric is invariant under the scaling

$$t \rightarrow \lambda^z t, \quad y_i \rightarrow \lambda^z y_i, \quad x_i \rightarrow \lambda x_i, \quad r \rightarrow r/\lambda. \quad (2.7)$$

The non-relativistic metrics we are interested in may also have a non-trivial r -dependent dilaton field Φ when embedded in the string theory, i.e. in practice in type-II supergravity. In those cases the factor e^Φ may have a monomial form, as it happens in many of the known supergravity solutions.

Another class of non-relativistic theories we will study correspond to the Schrödinger spacetime Sch_z , for which the metric can be written as

$$ds^2 = -r^{2z} dt^2 + r^2 \left(2dt d\xi + d\rho^2 + \rho^2 d\phi^2 + \sum_{i=3}^d dx_i^2 \right) + \frac{dr^2}{r^2}. \quad (2.8)$$

The isometry algebra of the Sch_z spaces contains the Galilean boosts as well as null translations of ξ . The latter play the rôle of the central extension or mass operator of the Galilean algebra. The presence of the off-diagonal term of the Sch_z space modifies the details of the approach for showing non-integrability compared to that for the Lif_z case. Hence we will analyze first the non-integrability of Lif_z spacetime and then that of the Sch_z one.

3 Non-integrability in generic non-relativistic spaces

In this section we first develop a general formalism and then we focus on a more applied analysis to the cases we have mentioned.

3.1 Consistency of sigma-model solutions

We first discuss general properties of the string solutions we construct after we transform to the string frame metric. These are localized in all directions except for a single cyclic coordinate which we allow to wind. We will show that our ansatz is consistent and that the Virasoro constraints do not contribute additional differential conditions to the system. They just constrain the allowed initial conditions.

Consider a general two-dimensional sigma-models of the form

$$S(X) = -\frac{\sqrt{\lambda}}{\pi} \int Q_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu, \quad Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}. \quad (3.1)$$

We use the coordinate conventions $x^\pm = \tau \pm \sigma$. The equation of motion for X^μ is

$$\ddot{X}^\mu - X''^\mu + \Gamma^\mu_{\nu\rho} (\dot{X}^\nu \dot{X}^\rho - X'^\nu X'^\rho) + H^\mu_{\nu\rho} \dot{X}^\nu X'^\rho = 0, \quad (3.2)$$

where $H_{\rho\mu\nu} = \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}$. The Virasoro constraints are given by

$$\begin{aligned} G_{\mu\nu} (\dot{X}^\mu \dot{X}^\nu + X'^\mu X'^\nu) &= 0, \\ G_{\mu\nu} \dot{X}^\mu X'^\nu &= 0. \end{aligned} \quad (3.3)$$

We will assume that there is a Killing isometry corresponding to constant shifts of the coordinate X^9 . We will let accordingly, the rest of the coordinates be denoted by X^i , so that the index $\mu = (i, 9)$. Consider next solutions of the form

$$X^i = X^i(\tau), \quad X^9 = m\sigma, \quad m \in \mathbb{Z}. \quad (3.4)$$

Then the equations of motion become

$$\begin{aligned} \mu = i : \quad & \ddot{X}^i + \Gamma^i_{jk} \dot{X}^j \dot{X}^k - m^2 \Gamma^i_{99} + m H^i_{j9} \dot{X}^j = 0, \\ \mu = 9 : \quad & \Gamma^9_{ij} \dot{X}^i \dot{X}^j - m^2 \Gamma^9_{99} + m H^9_{i9} \dot{X}^i = 0. \end{aligned} \quad (3.5)$$

Similarly, the Virasoro constraints become

$$\begin{aligned} G_{ij} \dot{X}^i \dot{X}^j + m^2 G_{99} &= 0, \\ m G_{9i} \dot{X}^i &= 0. \end{aligned} \quad (3.6)$$

By taking the τ derivative of the first Virasoro constrain and using the equations of motion we show that the constrain is preserved and therefore one just needs to satisfy them by imposing appropriate initial conditions. In addition, we assume that

$$G_{9i} = 0. \quad (3.7)$$

Then, the second equation of (3.5) is satisfied identically.

Therefore, for a theory satisfying (3.7) and for the particular string configuration (3.4), we are left only with the first equation of motion (3.5). An effective one-dimensional action from which this non-trivial equation can be derived from, is given by

$$S_{1\text{-dim}} = -\frac{\sqrt{\lambda}}{2} \int d\tau (G_{ij} \dot{X}^i \dot{X}^j - m^2 G_{99} + 2m B_{9i} \dot{X}^i) . \quad (3.8)$$

The Hamiltonian and conjugate momentum are given by

$$H_{1\text{-dim}} = -\frac{\sqrt{\lambda}}{2} (G_{ij} \dot{X}^i \dot{X}^j + m^2 G_{99}) = 0 , \quad P_i = -G_{ij} \dot{X}^j - m B_{9i} , \quad (3.9)$$

where setting $H_{1\text{-dim}} = 0$ implements the Virasoro constraint as in (3.6). In the next sections we consider supergravity solutions with zero B-fields. In several cases, our analysis can be applied even when B-fields are present. This is when the string configuration can be localized consistently such that no new contributions appear from the extra field. We elaborate further on that in the case of Schrödinger spaces.

3.2 Non-integrability of the string solutions

The set of backgrounds we will consider in this section include as special cases, the hyper-scaling violation metrics, as well as Lifshitz and Lifshitz-like backgrounds with possible non-trivial dilaton dependence. The metric

$$ds^2 = -r^\alpha dt^2 + r^\beta \left(d\rho^2 + \rho^2 d\phi^2 + \sum_{i=3}^d dx_i^2 \right) + r^\gamma dr^2 , \quad (3.10)$$

captures all of these cases by appropriately choosing the parameters α , β and γ .

We consider an extended string along a U(1) isometry corresponding to ϕ , i.e. $\phi = m\sigma$ and also take the x_i 's in the space (3.10) to be constant which obviously is consistent with the equations of motion. For the variables t , r and ρ we will allow only τ dependence. Using the results of the previous section, where the metric element $G_{99} := G_{\phi\phi}$, we find that the effective one-dimensional action (3.8) of the system is given by

$$S_{1\text{-dim}} = -\frac{\sqrt{\lambda}}{2} \int d\tau \left(-r^\alpha \dot{t}^2 + r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 - m^2 \rho^2) \right) . \quad (3.11)$$

The corresponding equations of motions are

$$\begin{aligned} t : \quad & \partial_\tau (\dot{t} r^\alpha) = 0 , \\ \rho : \quad & \partial_\tau (\dot{\rho} r^\beta) + m^2 r^\beta \rho = 0 , \\ r : \quad & \partial_\tau (2r^\gamma \dot{r}) + \alpha r^{\alpha-1} \dot{t}^2 - \gamma r^{\gamma-1} \dot{r}^2 - \beta r^{\beta-1} (\dot{\rho}^2 - m^2 \rho^2) = 0 \end{aligned} \quad (3.12)$$

and the vanishing Hamiltonian implementing the Virasoro constraints is

$$H_{1\text{-dim}} = -\frac{\sqrt{\lambda}}{2} \left[-r^\alpha \dot{t}^2 + r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 + m^2 \rho^2) \right] = 0 . \quad (3.13)$$

The equation of motion for t in (3.12) is solved for

$$\dot{t} = \frac{c}{r^\alpha}, \quad (3.14)$$

where c is the integration constant which can be taken non-negative with no loss of generality. Then, the remaining equations in (3.12) become

$$\begin{aligned} \ddot{\rho} + \beta \frac{\dot{r}}{r} \dot{\rho} + m^2 \rho &= 0, \\ c^2 \alpha r^{-\alpha-1} + \frac{1}{\dot{r}} \partial_\tau (r^\gamma \dot{r}^2) + \beta r^{\beta-1} (-\dot{\rho}^2 + \rho^2 m^2) &= 0. \end{aligned} \quad (3.15)$$

These equations can be derived from the effective one-dimensional action for the fields ρ and r , given by

$$S_{\text{eff}} = -\frac{\sqrt{\lambda}}{2} \int d\tau \left(r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 - m^2 \rho^2) + c^2 r^{-\alpha} \right). \quad (3.16)$$

The corresponding Hamiltonian is¹

$$H_{\text{eff}} = -\frac{\sqrt{\lambda}}{2} \left(r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 + \rho^2 m^2) - c^2 r^{-\alpha} \right). \quad (3.17)$$

Hence, we have consistently reduced the original string system to a classical one-dimensional Hamiltonian system for two fields ρ and r . A solution to the first equation in (3.15) is obtained for

$$\rho = 0. \quad (3.18)$$

Then, the equation for r and its solution are given by

$$\dot{r}^2 = c^2 r^{-\alpha-\gamma} \implies r(\tau) = \left(\frac{c}{2} (\alpha + \gamma + 2) \tau \right)^{\frac{2}{\alpha+\gamma+2}}, \quad (3.19)$$

where, with no loss of generality, we have set a constant of integration to zero. The solution is valid if $\alpha + \gamma + 2 \neq 0$. In the opposite case we comment appropriately below.

To obtain the NVE we perturb in the transverse plane so that $\rho = 0 + \eta(t)$, obtaining easily from the first of (3.15) that²

$$\ddot{\eta} + \beta \frac{\dot{r}}{r} \dot{\eta} + m^2 \eta = 0. \quad (3.20)$$

Using (3.19) to substitute for \dot{r}/r , we obtain that

$$\ddot{\eta} + \frac{1-2\nu}{\tau} \dot{\eta} + m^2 \eta = 0, \quad \nu = \frac{1}{2} - \frac{\beta}{\alpha + \gamma + 2}, \quad (3.21)$$

¹Note that H_{eff} below coincides with $H_{1\text{-dim}}$ in (3.13) after we substitute the solution (3.14). This is not the case at the level of the corresponding Lagrangians in (3.11) and in (3.16).

²One could also perturb the classical solution for $r(\tau)$ in (3.19). It should be obvious from the form of the system in (3.15) that the two fluctuations do not couple to linear order and therefore we may ignore the perturbation of r .

which has rational coefficients. Its general solution is given by

$$\eta(\tau) = \tau^\nu (c_1 J_\nu(m\tau) + c_2 Y_\nu(m\tau)) , \quad (3.22)$$

where J_ν and Y_ν are the Bessel functions of the first and of the second kind, respectively.

It is known that only when the order ν of the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ is half integer, then they are expressible in terms of elementary functions $\sin x$ and $\cos x$ divided by positive integer powers of \sqrt{x} . In that case, the NVE (3.22) admits Liouvillian solutions, the string solutions are integrable and the corresponding theory could be integrable. In all the other cases the Hamiltonian system, and the corresponding background is non-integrable. More explicitly the Liouvillian solutions appear for

$$\nu = \frac{2j+1}{2} \quad \implies \quad \frac{2+\alpha+\gamma}{\beta} = -\frac{1}{j}, \quad j \in \mathbb{Z} . \quad (3.23)$$

Therefore, when (3.23) is not satisfied for the metric form of (3.10) in string frame then the theory is not integrable. Notice that for $j = 0$, implying $\beta = 0$, (3.21) becomes the harmonic oscillator equation and the string configuration is integrable, independently of the value of α and γ .

Turning to the case with $\alpha + \gamma + 2 = 0$, the solution of (3.19) is simply $r = e^{c\tau}$. Then (3.21) becomes the harmonic oscillator with friction equation with solutions given by elementary functions. Hence the corresponding string is also integrable.

As a side remark notice that for the case of no string winding, i.e. $m = 0$, the resulting geodesics have always integrable structure.³ This is in agreement with the vast majority of the works in the literature, since non-integrability is more likely to be probed by extended strings.

3.3 Application to theories of special interest

3.3.1 Lifshitz non-integrability

Consider the Lifshitz metric

$$ds^2 = -r^{2z} dt^2 + r^2 \left(d\rho^2 + \rho^2 d\phi^2 + \sum_{i=3}^d dx_i^2 \right) + \frac{dr^2}{r^2} , \quad (3.24)$$

with a trivial dilaton. This is a particular case of the metric (3.10) with

$$\alpha = 2z, \quad \beta = 2, \quad \gamma = -2 . \quad (3.25)$$

Then the condition (3.23) gives

$$z = -\frac{1}{j}, \quad j \in \mathbb{Z} . \quad (3.26)$$

These are the only values for which we may have integrability. Specifically, the exact values, of z for which the Hamiltonian system is integrable are when

$$z \in \left\{ -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1 \right\} . \quad (3.27)$$

³When $m = 0$, the solution of (3.21) is a linear combination of a 1 and $\tau^{2\nu}$.

The null energy condition is satisfied for $z \geq 1$. Therefore the only physical Lifshitz theory that is integrable is for $z = 1$, i.e. the AdS background. Backgrounds with $z < 1$, where the dual field theories are not well defined, might be integrable for the values of z defined in (3.26).

Moreover, there are several interesting Lifshitz backgrounds in the literature with dilatons depending on the holographic coordinate r . For example, it has been found that the $AdS_5 \times S^5$ geometries backreacted by an $\mathcal{O}(N^2)$ density of heavy quarks, exhibit a Lifshitz geometry in the infrared with a dynamical critical exponent $z = 7$ and a dilaton having a $\ln r^6$ dependence on the holographic coordinate [44]. Applying the equation (3.23) we obtain $j = -1/4$ and therefore the theory is not integrable at least in the IR. The non-integrability of the theory may have even further implications to the observables of the theory, for example chaos may appear in some of them, and this worth a further investigation.

3.3.2 Anisotropic Lifshitz-Like non-integrability

It is not difficult to see that it is consistent to localize the string along the x_i directions of the anisotropic Lifshitz-like metric (2.6) by setting all the x_i 's to constant values. Then we may follow an identical analysis to the one corresponding to the metric (3.24) leading eventually to the conclusion that if (3.27) is satisfied then the corresponding string model may be integrable. For all other values integrability is excluded. Moreover, from the null energy condition the only physically allowed value allowing integrability is $z = 1$, corresponding to the AdS background.

A particular such anisotropic background is the holographic dual of Lifshitz-like fixed point found in type-IIB supergravity in [33], where a non-trivial dilaton exist. In the Einstein frame the metric is equivalent to (2.6) with $z = 3/2$, $k = 2$ and $d = 3$, while it has a non-trivial dilaton with $\ln r^{2/3}$ dependence. In the string frame it becomes equivalent to the metric (3.10) with $\alpha = \beta = 7/3$ and $\gamma = 5/3$ and the remaining anisotropic directions scale with a factor r^γ . However, they play no rôle in the analysis since the string is localized in there. Using the condition (3.26) we get the non-integer value $\nu = 8/7 \notin \mathcal{Z}$ and therefore the theory is not integrable.

3.3.3 Hyperscaling violation metrics non-integrability

Comparing the metrics (2.1) and (3.10) we identify the parameters

$$a = 2z - 2\frac{\theta}{d}, \quad \beta = 2 - 2\frac{\theta}{d}, \quad \gamma = -2 - 2\frac{\theta}{d}. \quad (3.28)$$

Using (3.23) we find that the corresponding NVE has Liouvillian solutions for the values

$$\frac{\theta}{d} = \frac{1 + jz}{1 + 2j}, \quad j \in \mathcal{Z}. \quad (3.29)$$

For any other values of θ and z , not satisfying the above conditions the corresponding theory is not integrable. By applying the null energy condition (2.4) and the one obtained

from entanglement entropy scaling (2.5) to ensure for physical dual gauge theories, we find that the necessary conditions for integrable physical theories to exist are

$$j = -1, \quad \theta = d(z - 1), \quad 1 \leq z \leq 2 - \frac{1}{d}. \quad (3.30)$$

For all other values of d , z and θ the theory is not integrable. The lower value $z = 1$ corresponds to the AdS_{d+2} case and the upper one $z = 2 - 1/d$ to gravity dual theories for Fermi surfaces. The saturation of these conditions happens among other values for $d = 2$, $\theta = 1$, $z = 3/2$ which occur for non-Fermi liquid states [43] and correspond to regular IR limit. We conclude that the majority of physical and non-physical theories is non-integrable.⁴

4 Geometries with non-diagonal terms

In this section we consider metrics of the form

$$ds^2 = -r^\alpha dt^2 + 2r^\delta dt d\xi + r^\beta \left(d\rho^2 + \rho^2 d\phi^2 + \sum_{i=3}^d dx_i^2 \right) + r^\gamma dr^2, \quad (4.1)$$

which is similar to (3.10), but with the extra off-diagonal term. In this class of backgrounds the Schrödinger geometries with trivial or non-trivial polynomial r -dependent dilaton are included as particular cases.

Proceeding similarly to section 3 we set the x_i 's to constants, $\phi = m\sigma$ and then we obtain the effective one-dimensional action

$$S_{1\text{-dim}} = -\frac{\sqrt{\lambda}}{2} \int d\tau \left(-r^\alpha \dot{t}^2 + r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 - m^2 \rho^2) + 2r^\delta \dot{t} \dot{\xi} \right), \quad (4.2)$$

for the remaining variables. The corresponding equations of motions are

$$\begin{aligned} t : \quad & \partial_\tau \left(\dot{t} r^\alpha - r^\delta \dot{\xi} \right) = 0, \\ \xi : \quad & \partial_\tau \left(r^\delta \dot{t} \right) = 0, \\ \rho : \quad & \partial_\tau \left(\dot{\rho} r^\beta \right) + m^2 r^\beta \rho = 0, \\ r : \quad & \partial_\tau (2r^\gamma \dot{r}) + \alpha r^{\alpha-1} \dot{t}^2 - \gamma r^{\gamma-1} \dot{r}^2 - \beta r^{\beta-1} (\dot{\rho}^2 - m^2 \rho^2) - 2\delta r^{\delta-1} \dot{t} \dot{\xi} = 0 \end{aligned} \quad (4.3)$$

and the vanishing Hamiltonian implementing the Virasoro constraint is

$$H_{1\text{-dim}} = -\frac{\sqrt{\lambda}}{2} \left[-r^\alpha \dot{t}^2 + r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 + m^2 \rho^2) + 2r^\delta \dot{t} \dot{\xi} \right] = 0. \quad (4.4)$$

The first of the equations in (4.3) are easily integrated once to give

$$\dot{t} = \frac{c}{r^\delta}, \quad \dot{\xi} = \frac{c}{r^{2\delta-\alpha}} - \frac{c_1}{r^\delta}, \quad (4.5)$$

⁴We do not consider negative values of the parameter θ or values for $d - 1 < \theta < d$. In case that the constraints of the physical theories will be relaxed to include some of these values and we end up with a slightly modified condition (3.30).

where c and c_1 are the integration constants. Then the remaining equations become

$$\begin{aligned} \ddot{\rho} + \beta \frac{\dot{r}}{r} \dot{\rho} + m^2 \rho &= 0, \\ \frac{1}{r} \partial_\tau (r^\gamma \dot{r}^2) + \beta r^{\beta-1} (-\dot{\rho}^2 + m^2 \rho^2) + c^2 (\alpha - 2\delta) r^{\alpha-2\delta-1} + 2\delta c c_1 r^{-\delta-1} &= 0. \end{aligned} \quad (4.6)$$

These equations can be derived from the effective one-dimensional action given by

$$S_{\text{eff}} = -\frac{\sqrt{\lambda}}{2} \int d\tau \left(r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 - m^2 \rho^2) - c^2 r^{\alpha-2\delta} + 2c c_1 r^{-\delta} \right). \quad (4.7)$$

The corresponding Hamiltonian is

$$H_{\text{eff}} = -\frac{\sqrt{\lambda}}{2} \left(r^\gamma \dot{r}^2 + r^\beta (\dot{\rho}^2 + \rho^2 m^2) + c^2 r^{\alpha-2\delta} - 2c c_1 r^{-\delta} \right). \quad (4.8)$$

Hence, we have consistently reduced the original string system to a classical one-dimensional Hamiltonian system for two fields ρ and r .

Choosing the same solution as in (3.18) we find that

$$\dot{r}^2 = c r^{-\delta-\gamma} (2c_1 - c r^{\alpha-\delta}), \quad (4.9)$$

which for generic values of α , β and δ has solution in terms of inverse hypergeometric functions. By varying the equation of motion for r with $\rho = 0 + \eta(t)$ we obtain the same form of the equation as in (3.20). Then using (4.9) we obtain

$$\eta'' + \left(\frac{2\beta - \delta - \gamma}{2r} + \frac{c(\delta - \alpha)}{2(2c_1 r^{\delta-\alpha+1} - c r)} \right) \eta' + m^2 \eta = 0, \quad (4.10)$$

where the derivative now is with respect to r . The NVE has rational coefficients when $\delta - \alpha + 1 \in \mathbb{Z}$, then the Kovacic algorithm can be applied. In contrary to the diagonal metrics, we do not find for generic values of α, β, γ and δ analytic solutions, so that we cannot tell when they become Liouvillian. Therefore to show non-integrability, we need to apply particular values to the parameters of our system, directly to the above NVE, as we do in the case of the Schrödinger geometries below. Notice that by localizing the string, i.e. choosing $m = 0$, we see that any geodesic of the type we have chosen is integrable, since the solutions to the above NVE are integrals of polynomials and square roots.

4.1 Application to Schrödinger geometries

The Sch_z geometries are obtained from the metric (4.1) by setting

$$\alpha = 2z, \quad \delta = 2, \quad \beta = 2, \quad \gamma = -2. \quad (4.11)$$

We find that for the values $z = 1, 2, 3$, then (4.10) has Liouvillian solutions, the corresponding strings are integrable and we can not exclude integrability of the theory. For $z = 4, 5, 6$ we have non-integrable systems. There are also higher values of z where integrability can not be excluded. For any value of z the test of integrability can be done directly by checking the type of solutions of (4.10). We find that for the vast majority of the z values our strings are integrable.

Notice that the $z = 2$ Schrödinger spaces can be constructed with an appropriate TsT transformation [45] and has been shown to be integrable [46–48] which is in agreement with our results. Indeed, our analysis can be applied directly to these backgrounds by localizing consistently the string configuration such that there is no contributions to the effective action by the antisymmetric tensor field which is present in these backgrounds.

5 Conclusions

In this paper we have used methods of classical integrability to study non-relativistic theories. Working in a general framework, our analysis boils down to a simple formula of the scalings of the metric elements which when is not satisfied integrability is excluded. This is achieved by analyzing some unique properties of the solutions of the differential NVE equations which are obtained from string configurations in spaces under study. As a next step we have applied our results to non-relativistic theories of special interest. In particular the physical Lifshitz theories with trivial dilatons are integrable only for the value $z = 1$ corresponding to the AdS geometry. Thus any other candidate theory of different scaling is excluded in terms of integrability. The same applies to the Lifshitz-like theories since the non-integrable string solutions we have found can be localized consistently in the additional anisotropic directions. In theories with hyperscaling violation scaling we have found that in the small range of physical theories for which integrability is not excluded, the Fermi-surface theories saturating the null energy condition, are included. We also performed the analysis for non-relativistic theories with non-diagonal metric elements. For these the study of integrability needed to be done case by case due to the absence of similar unique properties of the solutions of the corresponding NVEs.

For theories containing non-trivial dilaton, we brought their metrics to the string frame and applied our generic formulas. As an example, we explicitly checked the type-IIB anisotropic Lifshitz-like supergravity solutions of [33] and we found that it is not integrable.

For vanishing winding number, all of our geodesics are integrable. This means that in order to see the non-integrable structure of the model we need to probe it with extended strings. This is a common finding when using methods of classical integrability in the string theory models and is in some sense expected, since the richer structure of the string reveals the non-integrable structure of the theory. Note also that, in general, a chaotic motion implies non-integrable solutions but the opposite is not always true. In our solutions with the constraints set by the zero Hamiltonian, we detect no chaotic behavior, in the physical range of the parameters and coordinates. This finding does not necessarily exclude chaos in other range of parameters and energies or for other string solutions.

We also note that by excluding integrability in the non-relativistic spaces we have worked, we expect non-integrability to follow for a wide range of spaces that have as a particular limit these non-relativistic spaces. For example, for the geometries backreacted by a heavy quark density which in the IR limit flow to a Lifshitz geometry, we expect non-integrability.

It would be very interesting to investigate the implications of our findings to the observables of the dual NRCFT. We have found non-integrability in backgrounds which

are used to describe dual anisotropic plasmas, where a wide range of observables has been already investigated [36], backgrounds with heavy quark density and backgrounds used to describe condensed matter systems. Certain predictions of these non-integrable theories may be chaotic, like the hadronic spectrum in confining theories [23]. In our case would be interesting to investigate this possibility and the physical implications of the chaos to the observables.

Acknowledgments

We are thankful to P. Kumar, K. Siampos, A. Tseytlin and K. Zoubos for useful conversations, comments and correspondence. The research of the authors is implemented under the “ARISTEIA” action of the “operational programme education and lifelong learning” and is co-funded by the European Social Fund (ESF) and National Resources.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[INSPIRE](#)].
- [2] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys. B* **670** (2003) 439 [[hep-th/0307042](#)] [[INSPIRE](#)].
- [3] N. Beisert et al., *Review of AdS/CFT integrability: an overview*, *Lett. Math. Phys.* **99** (2012) 3 [[arXiv:1012.3982](#)] [[INSPIRE](#)].
- [4] S.L. Ziglin, *Branching of solutions and non-existence of first integrals in Hamiltonian mechanics I*, *Funct. Anal. Appl.* **16** (1982) 181.
- [5] S.L. Ziglin, *Branching of solutions and non-existence of first integrals in Hamiltonian mechanics II*, *Funct. Anal. Appl.* **17** (1983) 617.
- [6] A.T. Fomenko, *Integrability and nonintegrability in geometry and mechanics*, Kluwer Academic Publishers, U.S.A. (1988).
- [7] J.J.M. Ruiz, *Differential Galois theory and non-integrability of Hamiltonian systems*, Birkhauser, Basel, Switzerland (1999).
- [8] A. Goriely., *Integrability and nonintegrability of dynamical systems*, World Scientific, Singapore (2001).
- [9] A.V. Frolov and A.L. Larsen, *Chaotic scattering and capture of strings by black hole*, *Class. Quant. Grav.* **16** (1999) 3717 [[gr-qc/9908039](#)] [[INSPIRE](#)].
- [10] L.A. Pando Zayas and C.A. Terrero-Escalante, *Chaos in the gauge/gravity correspondence*, *JHEP* **09** (2010) 094 [[arXiv:1007.0277](#)] [[INSPIRE](#)].
- [11] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)] [[INSPIRE](#)].

- [12] P. Basu, D. Das and A. Ghosh, *Integrability lost*, *Phys. Lett. B* **699** (2011) 388 [[arXiv:1103.4101](#)] [[INSPIRE](#)].
- [13] P. Basu and L.A. Pando Zayas, *Chaos rules out integrability of strings in $AdS_5 \times T^{1,1}$* , *Phys. Lett. B* **700** (2011) 243 [[arXiv:1103.4107](#)] [[INSPIRE](#)].
- [14] P. Basu and L.A. Pando Zayas, *Analytic non-integrability in string theory*, *Phys. Rev. D* **84** (2011) 046006 [[arXiv:1105.2540](#)] [[INSPIRE](#)].
- [15] D. Giataganas, *Semi-classical strings in Sasaki-Einstein manifolds*, *JHEP* **10** (2009) 087 [[arXiv:0904.3125](#)] [[INSPIRE](#)].
- [16] D. Giataganas, *On the non-BPS string solutions in Sasaki-Einstein gauge/gravity duality*, *JHEP* **06** (2010) 016 [[arXiv:0912.3624](#)] [[INSPIRE](#)].
- [17] D. Giataganas, *Semiclassical strings in marginally deformed toric AdS/CFT*, *JHEP* **12** (2011) 051 [[arXiv:1010.1502](#)] [[INSPIRE](#)].
- [18] D. Giataganas, L.A. Pando Zayas and K. Zoubos, *On marginal deformations and non-integrability*, *JHEP* **01** (2014) 129 [[arXiv:1311.3241](#)] [[INSPIRE](#)].
- [19] Y. Chervonyi and O. Lunin, *(Non)-integrability of geodesics in D-brane backgrounds*, *JHEP* **02** (2014) 061 [[arXiv:1311.1521](#)] [[INSPIRE](#)].
- [20] R. de Mello Koch, M. Dessein, D. Giataganas and C. Mathwin, *Giant graviton oscillators*, *JHEP* **10** (2011) 009 [[arXiv:1108.2761](#)] [[INSPIRE](#)].
- [21] W. Carlson, R. de Mello Koch and H. Lin, *Nonplanar integrability*, *JHEP* **03** (2011) 105 [[arXiv:1101.5404](#)] [[INSPIRE](#)].
- [22] P. Basu, D. Das, A. Ghosh and L.A. Pando Zayas, *Chaos around holographic Regge trajectories*, *JHEP* **05** (2012) 077 [[arXiv:1201.5634](#)] [[INSPIRE](#)].
- [23] L.A. Pando Zayas and D. Reichmann, *A string theory explanation for quantum chaos in the hadronic spectrum*, *JHEP* **04** (2013) 083 [[arXiv:1209.5902](#)] [[INSPIRE](#)].
- [24] P. Basu and A. Ghosh, *Confining backgrounds and quantum chaos in holography*, *Phys. Lett. B* **729** (2014) 50 [[arXiv:1304.6348](#)] [[INSPIRE](#)].
- [25] A. Stepanchuk and A.A. Tseytlin, *On (non)integrability of classical strings in p-brane backgrounds*, *J. Phys. A* **46** (2013) 125401 [[arXiv:1211.3727](#)] [[INSPIRE](#)].
- [26] P. Basu and A. Ghosh, *Dissipative nonlinear dynamics in holography*, *Phys. Rev. D* **89** (2014) 046004 [[arXiv:1304.6349](#)] [[INSPIRE](#)].
- [27] A. Farahi and L.A. Pando Zayas, *Gravitational collapse, chaos in CFT correlators and the information paradox*, [[arXiv:1402.3592](#)] [[INSPIRE](#)].
- [28] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, *Phys. Rev. D* **78** (2008) 106005 [[arXiv:0808.1725](#)] [[INSPIRE](#)].
- [29] S.A. Hartnoll, J. Polchinski, E. Silverstein and D. Tong, *Towards strange metallic holography*, *JHEP* **04** (2010) 120 [[arXiv:0912.1061](#)] [[INSPIRE](#)].
- [30] K. Balasubramanian and K. Narayan, *Lifshitz spacetimes from AdS null and cosmological solutions*, *JHEP* **08** (2010) 014 [[arXiv:1005.3291](#)] [[INSPIRE](#)].
- [31] A. Donos and J.P. Gauntlett, *Lifshitz solutions of $D = 10$ and $D = 11$ supergravity*, *JHEP* **12** (2010) 002 [[arXiv:1008.2062](#)] [[INSPIRE](#)].

- [32] R. Gregory, S.L. Parameswaran, G. Tasinato and I. Zavala, *Lifshitz solutions in supergravity and string theory*, *JHEP* **12** (2010) 047 [[arXiv:1009.3445](#)] [[INSPIRE](#)].
- [33] T. Azeyanagi, W. Li and T. Takayanagi, *On string theory duals of Lifshitz-like fixed points*, *JHEP* **06** (2009) 084 [[arXiv:0905.0688](#)] [[INSPIRE](#)].
- [34] D. Mateos and D. Trancanelli, *Thermodynamics and Instabilities of a Strongly Coupled Anisotropic Plasma*, *JHEP* **07** (2011) 054 [[arXiv:1106.1637](#)] [[INSPIRE](#)].
- [35] D. Giataganas, *Probing strongly coupled anisotropic plasma*, *JHEP* **07** (2012) 031 [[arXiv:1202.4436](#)] [[INSPIRE](#)].
- [36] D. Giataganas, *Observables in strongly coupled anisotropic theories*, *PoS(Corfu2012)* 122 [[arXiv:1306.1404](#)] [[INSPIRE](#)].
- [37] S. Schäfer-Nameki, M. Yamazaki and K. Yoshida, *Coset construction for duals of non-relativistic CFTs*, *JHEP* **05** (2009) 038 [[arXiv:0903.4245](#)] [[INSPIRE](#)].
- [38] K. Goldstein, S. Kachru, S. Prakash and S.P. Trivedi, *Holography of charged dilaton black holes*, *JHEP* **08** (2010) 078 [[arXiv:0911.3586](#)] [[INSPIRE](#)].
- [39] K. Narayan, *On lifshitz scaling and hyperscaling violation in string theory*, *Phys. Rev. D* **85** (2012) 106006 [[arXiv:1202.5935](#)] [[INSPIRE](#)].
- [40] J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, *J. Symb. Comput.* **2** (1986) 3
- [41] X. Dong, S. Harrison, S. Kachru, G. Torroba and H. Wang, *Aspects of holography for theories with hyperscaling violation*, *JHEP* **06** (2012) 041 [[arXiv:1201.1905](#)] [[INSPIRE](#)].
- [42] L. Huijse, S. Sachdev and B. Swingle, *Hidden Fermi surfaces in compressible states of gauge-gravity duality*, *Phys. Rev. B* **85** (2012) 035121 [[arXiv:1112.0573](#)] [[INSPIRE](#)].
- [43] N. Ogawa, T. Takayanagi and T. Ugajin, *Holographic Fermi surfaces and entanglement entropy*, *JHEP* **01** (2012) 125 [[arXiv:1111.1023](#)] [[INSPIRE](#)].
- [44] S.P. Kumar, *Heavy quark density in $N = 4$ SYM: from hedgehog to Lifshitz spacetimes*, *JHEP* **08** (2012) 155 [[arXiv:1206.5140](#)] [[INSPIRE](#)].
- [45] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up galilean holography*, *JHEP* **11** (2008) 080 [[arXiv:0807.1099](#)] [[INSPIRE](#)].
- [46] D. Orlando, S. Reffert and L.I. Uruchurtu, *Classical integrability of the squashed three-sphere, warped AdS_3 and Schroedinger spacetime via T -duality*, *J. Phys. A* **44** (2011) 115401 [[arXiv:1011.1771](#)] [[INSPIRE](#)].
- [47] I. Kawaguchi and K. Yoshida, *Classical integrability of Schrödinger σ -models and q -deformed Poincaré symmetry*, *JHEP* **11** (2011) 094 [[arXiv:1109.0872](#)] [[INSPIRE](#)].
- [48] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Schroedinger σ -models and Jordanian twists*, *JHEP* **08** (2013) 013 [[arXiv:1305.6556](#)] [[INSPIRE](#)].